A Subtle Introduction to Category Theory

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> "Static concepts proved to be very effective intellectual tranquilizers."
>
> **L. L. Whyte**

> "Our study has revealed Mathematics as an array of forms, codifying ideas extracted from human activates and scientific problems and deployed in a network of formal rules, formal definitions, formal axiom systems, explicit theorems with their careful proof and the manifold interconnections of these forms...[This view] might be called *formal functionalism*."
>
> **Saunders Mac Lane**

Let's dive right in then, shall we? What can we say about the array of forms MacLane speaks of?

Claim (Tentative). Category theory is about the formal aspects of this array of forms.

Commentary: It might be tempting to put the brakes on right away and first grab hold of what we mean by formal aspects. Instead, rather than trying to present "the formal" as the object of study and category theory as our instrument, we would nudge the reader to consider another perspective. Category theory itself defines formal aspects in much the same way that physics defines physical concepts or laws define legal (and correspondingly illegal) aspects: it embodies them. When we speak of what is formal/physical/legal we inescapably speak of category/physical/legal theory, and vice versa. Thus we pass to the somewhat grammatically awkward revision of our initial claim:

Claim (Tentative). Category theory *is* the formal aspects of this array of forms.

Commentary: Let's unpack this a little bit. While individual forms are themselves tautologically formal, arrays of forms and everything else networked in a system lose this tautological formality. In the case of Mathematics, as Mac Lane states, we have ideas and latent/explicit physical metaphors and independent axiom systems, etc. that make Mathematics more than just one form. So Mathematics on the whole is not just formal, yet when we theorize its formal aspects we are doing category theory. In fact we can use category theory to study the formal aspects of anything: physics, logic, politics, programming, psychology, etc. We will call the "formal aspects of something" its *structure.*

Claim. Category theory is structure.

I hope you'll bear with me while I endeavor to support this claim. We'll have fun along the way, and, of course, remember that it is the journey, the transition, the process, that counts. Not the beginning (we're already there) and it is certainly not just the end. What counts is the change that takes us from one to the other. This aphorism functions as an polite doorman to usher us into our next section.

1 What is a Category?

> "The universe is change..."
>
> **Marcus Aurelius**

1.1 A literate aside

We will find that a category is an abstract concept built up from an abstract idea of change. Taking a moment to briefly sympathize with the single-celled narrator of one of Italian writer Italo Calvino's short stories might well wet our palate:

> "Let's begin this way, then: there is a cell, and this cell is a unicellular organism, and this unicellular organism is me, and I know it, and I'm pleased about it. Nothing special so far. Now let's try to represent this situation for ourselves in space and time. Time passes, and I, more and more pleased with being in it and with being me, am also more and more pleased that there is time, and that I am in time, or rather that time passes and I pass time and time passes me, or rather I am pleased to be contained in time, to be the content of time, or the container, in short, to mark by being me the passing of time."
>
> Qfwfq in *Mitosis* by **Italo Calvino**

This is a strong sentiment. Qfwfq feels his very existence ("being me") consists of changes: marks in the passing of time. His identity is phenomenologically caught up in changes in time.

But we aren't just concerned with the identities of people, who seem to think and exists in/through/with time. What about the identities of things, especially non-physical things like justice or abelian groups? Can we abstract our idea of change away from the seeming physicality of time? We can refer to another story, where, rather than trying to describe the experience of the first cell, Qfwfq attempts to describe the first thing, the first sign. He says:

> What sort of sign? Its hard to explain because if I say sign to you, you immediately think of a something that can be distinguished from a something else, but nothing could be distinguished from anything there. (Calvino, Cosmicomics 31)

That's the idea! Changes in time are but one instance of abstract differences, and importantly we understand abstract differences as *processes* of distinguishing something from something else. With these fundamentals hinted at, let's take our time being a hell of a lot more precise in the next section.

1.2 Defining a Category

The basic notion in a category is that of (changes/processes/creations of difference) that are technically called morphisms. We give a recursive definition:

Definition 1. *A* **morphism** f *consists of two morphisms:*

1. *the source morphism* $s : f \to A$ *and*
2. *the target morphism* $t : f \to B$.

We refer to A *as the* domain *of* f *and* B *as the* codomain *of* f*, using the notation* $f : A \to B$.

Commentary: In this recursive definition we are embodying the structure of (changes/processes/creations of difference). In regular English the sentences are a bit obtuse, but we can give versions for both temporal and nontemporal instances respectively:

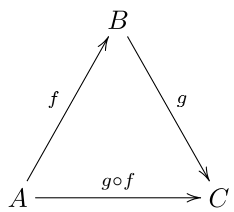
1. We can change a given change into either what we begin with or what we end with when we perform that given change.
2. Given a difference we can differentiate something from something else.

Our definition of a morphism gives structure common between these two. Note that in some literature, and sometimes here, morphisms are called arrows, since drawn arrows represent them well.

Definition 2. *If the domain of a morphism* $g : B \to C$ *is the codomain of* $f : A \to B$*, then a* **composite morphism** *is defined as*

$$(g \circ f) : A \to C \tag{1}$$

Commentary: It is often helpful to represent morphisms graphically in what is called a "commutative diagram"



Commutative diagrams state that any path you take along the arrows (composing them end to end with $\circ$) is equivalent to any other. This is a trivial commutative diagram which simply states that $g \circ f = (g \circ f)$.

Following our examples that instantiate things dynamically and statically, examples of the structure of composite morphisms are:

1. the f change and then the g change
2. that distinguishing a first thing from a second and a second from a third will also distinguish the first from the third.

Definition 3. Identity morphisms *are defined as morphisms* $1_A : A \to A$ *such that for any morphism* $f : A \to B$

$$f \circ 1_A = f \tag{2}$$

and for any morphism $g : B \to A$

$$g = 1_A \circ g \tag{3}$$

These are respectively known as the right and left identity properties.

Even with as little mathematics as has been presented we can investigate the consequences of the above structure:

Theorem 1. *Identity morphisms, if they exist, are unique.*

Proof. If there are two identity morphisms $1_A : A \to A$ and $e_A : A \to A$, then they are equal.

$$\begin{aligned} e_A &= e_A \circ 1_A && \text{by the right identity of } e_A \\ &= 1_A. && \text{by the left identity of } 1_A \end{aligned}$$

□

We now have enough to define a category in full.

Definition 4. *A* **Category** $\mathcal{C}$ *consists of morphisms such that*

1. *for every domain* A *and codomain* B *there are identity morphisms* 1_A *and* 1_B*.*
2. *there are all composite morphisms*
3. *composition is an associative operation, i.e. given morphisms* $f : A \to B$, $g : B \to C$, *and* $h : C \to D$ *the following composites are equal*

$$h \circ (g \circ f) = (h \circ g) \circ f : A \to D \tag{4}$$

where parentheses notate the order of composition.

Since the identity morphisms are in one-to-one correspondence (due to the uniqueness proof above) with the domains and codomains, we will often refer to them as the *objects* of a category, as opposed to other non-identity morphisms. This leads us to the traditional presentation of a category as objects with arrows (morphisms) between them. It was important to introduce objects at this later stage to highlight how they are not independent of morphisms but rather are just a particular kind of morphism.

2 Some Explicit Examples of Categories

It is hard to overstate how strong the case is for the idea that categories are everywhere, so, while there is a lot more to category theory than just the basic definition of a category, there are already plenty of examples worth illustrating to reinforce this definition. In the following sections we will look at the structure of numbers through finite categories, symmetries through groups and groupoids, membership through sets, logic through the category of proofs, and at some other examples of categories found throughout mathematics.

2.1 Number: Finite Categories

We introduce one diagrammatic representation of categories (called quivers), which is to write their objects (the identity morphisms) as vertices in a graph with arrows pointing from domains to codomains representing the morphisms. Quivers also relax the requirement of closure under composition, as in the diagrams we leave some morphisms generated by composition implicit. These first few examples should give you the hang of it.

Example 1. The category **0** has no morphisms.

Commentary: This is the formal aspect of Qfwfq's first sign and we notice that it is completely empty. There is nothing to compare it to, and it has no structure of its own. Claim: Nothing is **0**.

Example 2. The category **1** is a single morphism $1_* : * \to *$. This is the same as saying that **1** has a single object $*$. A diagram for this category is correspondingly simple:

$$*$$

Commentary: This category structures "the point" by a single relation. The structure of symmetric Self (domain) and Other (codomain) that defines a single identity: **1**.

Example 3. The category **2** is three morphisms: two identity morphisms (objects) and a third morphism $f : A \to B$ between them. This is diagrammed as

$$\mathrm{A} \longrightarrow \mathrm{B}$$

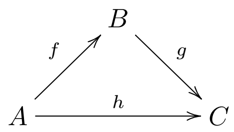
We can easily check that this meets the requirements for a category given in Definition 4:

1. There is only one domain and codomain and each has an identity morphism.
2. We use the definitions of identity morphisms to show that all composite morphisms are already in the category. Another way to note this is to say that these three morphisms are *closed under composition*. There are two possible compositions: $f \circ 1_A$ and $1_B \circ f$. Using the definitions of identities 1_A and 1_B we know that $f \circ 1_A = f$ and $1_B \circ f = f$, which is indeed a morphism already in the category.

3. It can be similarly shown that the definitions of identities also mean composition is associative.

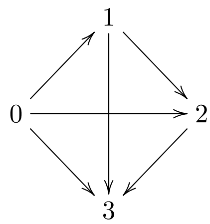
Commentary: The claim here is that to understand two (for two to formally and functionally exist), we use three relationships with the structure of **2**.

Example 4. The category **3** is six morphisms: three identity morphisms (objects) and three morphisms $f : A \to B, g : B \to C, h : A \to C$ between them. We've seen a similar diagram before



Remembering the requirement that every composition of morphisms must be in the diagram, we recognize that it must be the case that $h = g \circ f$.

Example 5. The category **4** is ten morphisms with the following diagram. Here we will label the objects with numbers for suggestive reasons.



What these examples of finite categories are beginning to define is the structure of ordinal numbers. Ordinal numbers are numbers that describe the position of something in a sequence: first (**1**), second (**2**), third (**3**), fourth (**4**), and so on.[1]

Example 6. The category **n** is $n(n+1)/2$ morphisms that follow the pattern set out above and structures ordinal numbers.

Commentary: This means that these categories give the structure for anything that is ordered and so counted in that order.[2]

[1] In set theoretic mathematics this structure is that of a well ordered set, where we define a binary relation allowing us to compare pairs and rank one above the other.

[2] Phenomenologically it might seem that all differences are ordered in that we apprehend them in a time ordering, but in fact it is only *when we count them* that we order differences. Before that a simple analogy could be that we process in parallel, but we don't need to rely on vague analogies. In other words, it's good that we have structures other than just finite categories (ordinal numbers) to talk about or we could get mired in pseudo-metaphysical confusions that simply arise from not having a general enough structural understanding of number.

2.2 Properties: Concrete Universals

To provide a first brief example of a non-mathematically bound category we will look to how categories structure concrete universals in philosophy.

Example 7. The category of properties **Prop** has things as objects and property relations as morphisms.

Commentary: What we mean by property relations is that the morphism $f : A \to B$ means A has a property of B. We could read an arrow between A and B as "A shares a property with B". As an example, consider "a white picket fence", which a thing and so is an object in **Prop**. There is a property relation (morphism) from "white picket fence" to "white paper" and a property relation from "white paper" to "whiteness." Clearly, there is also a composed property relation from "white picket fence" to "whiteness" and it is also clear that such relations will be associative. In (some) philosophy, a universal perfectly exhibits a particular property. For example, whiteness is that which is perfectly white. Category theory structures universals as *universal constructions*, one example of which is as limits, as processes of eliminating imperfections to approach a universal. We won't go into the definitions of universal constructions right now, but will merely point out that universals constructed in **Prop** are necessarily *concrete* universals.

The reason **Prop** structures concrete universals, is that all objects in **Prop** (including universals such as "whiteness") have themselves as properties, as given by their identity morphism.[3] This contrasts with *abstract* universals that are not properties of themselves and that are structured by categories of sets with subset inclusions as morphisms.[4]

Note 1. Let's take a moment to remember that in and of itself this doesn't say anything about whether this philosophical position is true or even reasonable. What is does say is that if you want to work consistently within this view (with all its entailments, such as that "whiteness" is a thing), then **Prop** is the structure for you.

Example 8. In critical theory, there is much debate over ideas of universality and hegemony and it wouldn't be a bad idea to look at these discussions structurally. Just as a trite and self-contained example let's take the following claim that is made on Wikipedia regarding Butler's criticism of Laclau's in [1]:

> "In the psychoanalytic theory of Jacques Lacan, 'the Real' is regarded as the limit of representation. Laclau draws upon this concept of the

[3] In fact, remembering the contingency of objects in our definition for a category, they *are* identity morphisms.

[4] Set theory deals with abstract universals to avoid Russels paradox, which emerges with regard to the set of all sets that are not sets of themselves. This paradoxical set an only exist if "set-ness" is a concrete universal. As we can build set theory within category theory, category theory can structure both abstract and concrete universals.

> Real to justify his claim that political identities are incomplete. Butler criticizes this because, according to her, it elevates the Lacanian Real into a transcendental, ahistorical category. However, Laclau's response to Butler on this point is that the Lacanian Real introduces a radical disjunction into our idea of history - something which puts the whole idea of a concept being 'ahistorical' radically into question (66). In other words, for Lacan, there is no continuity to history and, therefore, there can be no stable 'ahistorical' concepts."

Commentary: The idea of the Real as a limit of representation clearly makes it a universal.[5] Thus Butler's criticism of Laclau could be that Laclau is claiming that the Real is an abstract universal, implying that he is working in some category of history **Hist** (with events as objects and causality as morphisms) that has a set-like structure. This is the structure that would be necessary to have abstract historical universals. There would some fascinating ways to investigate the consequences of seeing history in this way, e.g. do we mean that there exists a certain kind of functor (see Section 3)

$$U : \mathbf{Hist} \to \mathbf{Set} \qquad ?$$

Laclau's response is that the idea of the Lacanian Real instead questions the presupposition that we can structure **Hist** as a category at all, since without continuity we lose closure under composition of causalities. Such a demonstration would also be interesting to see exhibited.

2.3 Symmetry: Groups

Other examples of categories are group-like categories called groupoids where we develop the notion of inverse processes or symmetric relations.

Definition 5. *A morphism $f : A \to B$ is* **invertible** *if there is a two-sided inverse $f^{-1} : B \to A$ such that both*

$$f^{-1} \circ f = 1_B \qquad f \circ f^{-1} = 1_A$$

Commentary: This embodies the structure of the idea that we can undo certain processes exactly, as if nothing happened. That is, there is a symmetrical relationship between a process and its inverse.

Definition 6. *Two objects are* **isomorphic** *if there is an invertible morphism between them.*

Commentary: We can freely and losslessly translate back and forth between two isomorphic objects

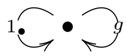
Definition 7. *A* **groupoid** *is a category where every morphism is invertible.*

[5] We could exhibit the categorical structure for this, but will choose not to in this treatment.

We will first consider groupoids that have only a single object, mathematically these are referred to as groups.

Note 2. On notation: Groups are well studied mathematical objects and many of them have their own names, such as the n element cyclic groups $\mathbb{Z}_n$. Thus when we look at the categorical structure of these objects we will denote the category corresponding to a group G as **BG**. **BG** is sometimes called the delooping of the group G.

Example 9. The category $\mathbf{B}\mathbb{Z}_\mathbf{2}$ is an identity morphism $1_\bullet : \bullet \to \bullet$ and a morphism $g : \bullet \to \bullet$. We can diagram it explicitly as:



Commentary: This gives the categorical structure of the smallest non-trivial group. Let's take a look at what the requirement that morphisms be closed under composition means for this category. Consider $g \circ g$, which is an allowable composition (it is "well typed") since g has the same domain and codomain. To keep closure either $g \circ g = g$ or $g \circ g = 1_*$. It must be the latter, since if $g \circ g = g$ then we know that g is an identity morphism on $*$ and so, by Theorem 1, $g = 1_*$ and the category would not be distinct from **1**. Thus $g \circ g = 1_*$. In this way g is sort of half an identity, as doing it twice is equivalent to doing nothing. This vague half-identity notion is formalized in the following notions of groups (which for two elements are all equivalent):

symmetric group, i.e. all the permutations of two symbols
cyclic group, i.e. alternating, the oscillation from one to another and back. Even's and odd's, multiplying by -1, off and on processes, clock ticks, etc.
dihedral group, i.e. the group of symmetries (reflections and rotation) of a polygon with two vertices (just one side).[6]

all of whom's structure is given by the two morphisms of the category $\mathbf{B}\mathbb{Z}_\mathbf{2}$.

Example 10. The category $\mathbf{BS_3}$ or the delooped dihedral group of degree three is given by six morphisms $1_\bullet, a, b, x, y, z : \bullet \to \bullet$. The quiver diagram is not particularly illustrative so we won't reproduce one. These morphisms are geometrical processes of rotations and reflections of an equilateral triangle that leave the triangle invariant. They are shown in Figure 1. We can describe them as follows:

$1_\bullet$ Do nothing.
a is a 120 degree rotation.
b is a 240 degree rotation.
x is a reflection around a x axis.
y is a reflection around a y axis.
z is a reflection around a z axis.

The morphisms in $\mathbf{BS_3}$ obey rules of composition according to Table 1.

[6] This is sort of a trivial case of the dihedral group.

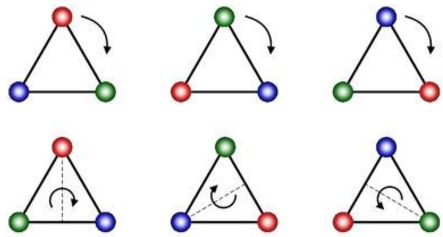


Fig. 1. The symmetry operations on an equilateral triangle listed from left to right starting with the top row: $1_\bullet, b, a, x, y, z$.

	$1_\bullet$	a	b	x	y	z
$1_\bullet$	$1_\bullet$	a	b	x	y	z
a	a	b	$1_\bullet$	y	z	x
b	b	$1_\bullet$	a	z	x	y
x	x	z	y	$1_\bullet$	b	a
y	y	x	z	a	$1_\bullet$	b
z	z	y	x	b	a	$1_\bullet$

Table 1. Table of composition equivalences for $\mathbf{BS_3}$. Choose a morphism, find its row, and then read off the results of its composition with the other morphisms by looking through the row.

Commentary: This category clearly structures our geometric intuition behind the rotation and reflection symmetries of an equilateral triangle. It is also the category for the delooped symmetric group of order three, or all the permutations of three different objects. These two notions have the same structure. $\mathbf{BS_3}$ is also notable as the smallest group (least number of morphisms) for which the order of composition matters, i.e. it is not always the case that $f \circ g = g \circ f$. This is what we mean when we call it the smallest *non-Abelian* group. Even when each individual process can be exactly undone (as is the case for all groupoid morphisms) we can still have structures where the order of processes is crucial.

One way of looking at these groups is that they structure symmetries as ways that a thing can be isomorphic to itself. Groupoids thus generalize groups to allow us to parameterize different ways that things can be isomorphic, one way for each object in the groupoid. In this way groupoids can be viewed as a generalized equivalence relation.

Note 3. The idea that the structure of symmetry is that of groups (and their actions) initial stemmed from the Erlanger program of Felix Klein and further work by Sophus Lie where groups of automorphisms give geometric structure. See [3] for a description. A larger class of symmetries are then characterized by their generalization to groupoids. See [2] for summary and description of the connection between groupoids and symmetry.

2.4 Membership or Collection: Sets

This section appears to have too many open questions for me to sort it all out right now, but here, I think, is a bit of a circular current state of things. What follows is certainly not accepted mathematical dogma...yet.[7]

If you aren't using sets as your foundation for mathematics, than you can at least be sure they are very close. We will define a few structures and then wave our hands a little to nudge towards a category theoretic structure for a set.

Definition 8. *A* **discrete** *category is a category with only identity morphisms.*[8]*.*

Commentary: This notion of separateness, or distinctness, of objects is clearly different than one from counting/ordering them using finite categories. All the objects are so different that there is no relation between them at all, except their membership in the same category.[9]

Definition 9. *A* **skeletal** *category is a category where all isomorphic objects are exactly equal.*

Commentary: In other words, if we are operating in a skeletal category then all morphisms between different objects have irreversible effects, i.e. they cannot be perfectly inverted. This also means that we lose track of operations that can be perfectly undone, such as changing coordinates or units of measurement, but that are sometimes a good idea to keep track of.

Definition 10. *A* **small** *category has objects and morphisms that can be represented as sets.*

Commentary: Clearly circular. The idea here is that the set of all sets is a large set and not a small set and so we can get rid of it by thinking only about small sets.

Definition 11. *A* **set** *is a small, discrete, skeletal category.*

Commentary: The metaphors involved here are, in fact just that: involved, so I'll leave it for now as open space in need of technical cleaning and certainly tons more explication. "What a set is" is certainly an open question.

Note 4. We will return to the notions that axiomatize sets and set theory later, but we will need much more of the machinery of category theory in order to structure what is going on. This isn't so surprising though. At some level the right notion of set theory is all you need to do (almost) all of mathematics! So first we will require the definition of terminal objects, pullbacks, exponents, universal constructions, initial objects, and elementary topoi, at least. We'll revisit this topic later.

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[7] This might be true of more that is in these notes than I thought.

[8] Sometimes this is also used to describe categories that are *categorically equivalent* to a category with only identity morphisms, but this idea goes beyond the scope of this example

[9] It seems like some morphisms could be constructed here, even if they like strictly outside of the discrete category in question. Inclusion functors between the objects of the category and the category itself?

2.5 (Classical) Logic: Proofs

Example 11. Given a deductive system of logic[10], a **category of proofs** consists of formulas as objects and deductions as morphisms.

Commentary: We certainly believe that deductions are associative and can be composed, and the identity morphism is trivially present. I can't think of a sensible deductive system where a given formula X does not imply X. In fact there are many useful relationships between different types of categories and different types of logics:

Regular categories structure regular logic.
Coherent categories structure coherent logic.
Heyting categories structure intuitionistic logic.
Boolean categories structure classical logic.
Monoidal categories structure linear logic.

2.6 Categories in Mathematics

While individual sets can be a little tricky to understand categorically, we can take the notion of a set as defined and use category theory to then study them. In fact, sets on the whole form a category!

Example 12. The category **Set** has (small) sets as objects and functions as morphisms.

Commentary: **Set** is not alone in this regard. In fact many mathematical objects can be understood as objects in categories with structure preserving maps between them. What follows are more basic examples.

Example 13. **Grp** consists of different groups as objects and group homomorphisms as morphisms. **Ab** consists of Abelian groups with group homomorphisms as morphisms.

Commentary: Not only are particular groups themselves each a category, but groups generally form a category.

Example 14. **Vect** consists of vector spaces[11] as objects and linear transformations as morphisms.

Commentary: Anything in linear algebra can be understood as operations in the proper category. For example, coordinate transformations in classical physics are morphisms. This applies equally well to other fields where vectors spaces play an important role, such as statistical analysis, denotational semantics in linguistics, etc.

[10] This originated with Lambek

[11] **Vect** is usually taken to have objects that are vector spaces over the real numbers, but there are also other categories whose objects are vector spaces over other fields, such as $\mathbb{C}$-**Vect**.

Example 15. **Top** consists of topological spaces as objects and continuous maps as morphisms.

Generally speaking, anything that mathematics has been applied to is somewhere we can find examples of categories.

It is interesting to notice the perspective that underlies this approach to organizing mathematics. We think of mathematical structures not only as their definitions, but instead, crucially, as consisting of the structure preserving morphisms between them. What is set-like about sets? That which is preserved by functions. What is topological about topological spaces? That which is invariant under continuous maps. This dynamical and relational quality is a fundamental part of thinking systemically with category theory. It's very systems theoretic:

> "When Bateson says that information is the difference that makes a difference, he is referring to that use of distinction, within any given set of variables, which makes the further and continued transformation of difference (e.g. reproduction) possible." **Antony Wilden** in *System and Structure* 222

These ideas of invariants are very, very important and are embodied in this allegiance to structure preserving morphisms (and morphisms in general) being a structural building block.

3 Metaphors and Meaning: Functors

The importance of structure preserving morphisms extends to categories themselves, and historically was one one of the main reasons they were developed. What we're saying is that categories themselves form a category, with particular morphisms called functors as their structure preserving maps.

Example 16. The category **Cat** has categories as objects and functors as morphisms.

Definition 12. *A* **functor** *from a category* $\mathbf{C}$ *to* $\mathbf{D}$ *is a morphism* $F : \mathbf{C} \to \mathbf{D}$ *which assigns to each object* C *of* $\mathbf{C}$ *an object* FC *of* $\mathbf{D}$ *and to each morphism* $f : C \to C'$ *in* $\mathbf{C}$ *a morphism* $Ff : FC \to FC'$ *in* $\mathbf{D}$ *such that both*

1. $F1_C = 1_{FC}$
2. $F(g \circ f) = (Fg) \circ (Ff)$*, for any* $g \circ f$ *in* $\mathbf{C}$*.*

That is, functors preserve identities and composites: the structure of categories. Figure 2 provides a diagram of the sort of thing that is going on.

Commentary: This notion allows us to equate meta-category-theory with category theory. As an example we can clarify some of the structures used so far.

Example 17. Forgetful functors go from one category to one with less structure. For example, there is a forgetful functor from **Grp** (the category of groups) to the category **Set**. This functor maps each delooped group to the set of its morphisms, "forgetting" everything about the compositional structure.

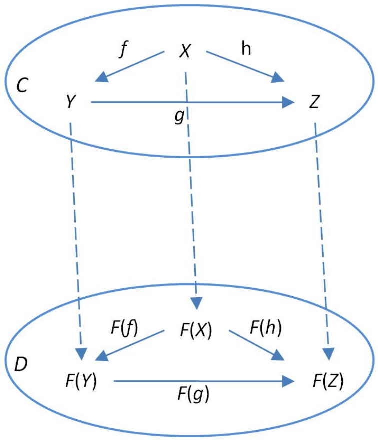


Fig. 2. The source and target categories are represented by the planes labeled C and D. The behavior of the functor (dotted line) $F : \mathbf{C} \to \mathbf{D}$ is to represent the category $\mathbf{C}$ in $\mathbf{D}$ in a way that preserves the categorical structure of $\mathbf{C}$.

Commentary: Of course this isn't a precise definition, but rather a guiding rule of thumb.

Example 18. We used quivers in Section 2.1 to allow us to diagram categories. We can now introduce quivers rigorously and categorically as follows. We assert that there exists a forgetful functor from **Cat** to the category of quivers (and structure preserving maps between them) which we notate as **Quiv**:

$$U : \text{Cat} \to \text{Quiv}$$

that takes each object of a category to a vertex in a graph and takes only the basic morphisms of the category to directed edges in the graph. We "forget" the morphism-ness of objects and forget compositions of morphisms.

Commentary: This is certainly a very simple construction, but it is illustrative of the way that seemingly meta-categorical ideas themselves exhibit categorical structure. Also, once we have a forgetful functor, we can construct another functor in the opposite direction called the "free functor" that maps from each quiver to the category it represents. The relationship between this free and forgetful functor is that of an *adjunction*. Adjunctions are dual morphisms in 2-categories, which we won't get into, but points to the fact that the notion of free and forgetful functors used here are in no way structurally empty notions even if they seem simple. And they shouldn't be simple, in this example we have used them to come up with an intuitive and accurate diagrammatic representation of categories, which *a priori* isn't such a simple task.

Example 19. Given a functional programming language L, there is a program category $\mathbf{C_L}$, where the objects are the data types of L, and the morphisms are programs written in L. We can clearly use the output of one program on the input of another in a way that is associative, and the program that simply returns its input is the identity program. The denotational semantics (or the mathematical representation of the programming language) is given by functors from $\mathbf{C_L}$ to other mathematical categories.

Commentary: Functors provide an insight into semantics in general, as we can see in the following example.

Example 20. Semantics are functors between categories of signifiers and categories of denotations.

Example 21. In quantum physics, particles are projective group representations of symmetry groups. Categorically, this means that particles are functors

$$P : \mathrm{BG} \to \mathrm{Hilb}$$

from a delooped symmetry group category to the category of Hilbert spaces (special kinds of vector spaces).

Commentary: Conceptually this says that particles are representations of the symmetries of the universe in another kind of mathematical object. In much the same way that diagrams of quivers allow us to understand given categories, particles allow us to formulate physical symmetries in terms of the vector spaces that we use in quantum theory.[12]

Definition 13. *Given a category C, it's* **opposite category** C^{op} *is given by a functor that does nothing to objects and takes*

$$(f : A \to B) \mapsto (f' : B \to A).$$

Example 22. In social choice theory, given alternatives structured by an object A, we consider the category $\mathbf{C_A}$ whose objects are subobjects of A with morphisms that are injective maps. Preferences are then understood as a functor

$$\mathbf{P} : \mathbf{C_A}^{\mathrm{op}} \to \mathbf{Set}$$

Generally speaking, when we have a functor we have the ability to translate one categorical structure into another. These are the most rigorous and powerful metaphors. And, of course, there are metaphors of metaphors, i.e. one can also construct functor categories, where the objects are functors and the morphisms are called *natural transformations* that preserve the structure of functors. In the case of physics, what matters for particles is the kind of vector space that their functor maps into, not the arbitrarily chosen basis or coordinates of that space. Basis transformations (coordinate changes) are then natural transformations on the particles, leaving their structure unchanged.

[12] Topological Quantum Field Theories are also functors, from cobordisms to vector spaces, but there's no reason to get fancy just because they are elegant structures.

So far we have only defined two big concepts: categories and functors. Yet together they structure an enormous variety of, well, very generally, *things*. And, importantly, we have left some insights into how they structure themselves. I'd argue that there is a lot to be gained by the recognition of categorical structure. It allows for the rapid and rigorously accurate translations of concepts into other fields with other categories that could yield unexpected and creative expressions. Even in just the examples above, we see the sense in which particles are the semantics in one part of physics and preferences the semantics of choice, letting us more precisely understand what we mean when we say that physics is a language or meaning is about choice. In this way category theory is not only organizationally elegant, but a powerful tool for any constructive art.

There is just so much more that could be said about structures for basic categories, especially regarding universal constructions, natural transformations, adjoints, exponents, monoidal categories, categorical products, duals, adjoints, graphical languages for categorical reasoning, and more and more and more. And a lot of it might be on the way; it's somewhere between the literature and my head.

4 Monoidal Categories

Monoidal categories add an additional structure on top of basic categories that allows us to acknowledge two different types of composition. One way of motivating monoidal categories is as a way of structuring physics.

Example 23. The category of physical process **PhysProc** has

- physical systems as objects, with the the "do nothing" process as the identity morphism
- physical processes as morphisms (these processes often occur over some time interval)
- sequential composition of processes in time as morphism composition

Commentary: This is a general category to work in for describing physics, but there seems to be more structure to **PhysProc** than that of just a basic category: there is an idea that we can have two physical systems composed in parallel, i.e. there is composition in space as well as in time[13] This leads us to the definition of a strict monoidal category. First we will take a look at a few other definitions from category theory.

Definition 14. *A* **hom-set** *$hom(x, y)$ is the collection of all morphisms with object x as domain and y as codomain.*

[13] This space-like and time-like severation is simply a motivating example as the structure of a monoidal category is more general than that of our physical notions of Newtonian/Einsteinian/Godelian/etc space and time. This is good as we want to go beyond these specific theories.

Commentary: $\text{Hom}(x, y)$ gives all the arrows between x and y.[14]

Definition 15. *A* **Monoid** *is a category with a single object.*

Commentary: Monoids are groups where we have relaxed the requirement that every morphisms be invertible. We still have an identity morphism and can compose the morphisms in a monoid with results that will always be closed within the monoid. Monoids are referred to as triples $(M, \otimes, I)$ where M is the set of morphisms, $\otimes$ is the symbol for morphism composition, and I is the symbol for the identity morphism.

Note 5. For a small category $\mathbf{C}$ we use the notation $\text{Ob}(\mathbf{C})$ to mean the set of objects of $\mathbf{C}$.

Definition 16. *A* **strict monoidal category C** *is a category for which,*

1. *The objects of the category have a monoid structure $(Ob(\mathbf{C}), \otimes, I)$, i.e. for objects A, B, C there is a monoidal composition $\otimes$ and unit I such that*

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C \qquad \text{and} \qquad I \otimes A = A = A \otimes I$$

This means we have objects not only like A and B, but also $A \otimes B$.

2. *The morphisms of $\mathbf{C}$ also have a monoid structure with a composition*

$$- \otimes - : hom(A, B) \times hom(C, D) \to hom(A \otimes C, B \otimes D), \text{ such that } (f, g) \mapsto f \otimes g$$

which is associative and whose unit it the identity morphism 1_I, i.e.

$$f \otimes (g \otimes h) = (f \otimes g) \otimes h \qquad \text{and} \qquad 1_I \otimes f = f = f \otimes 1_I$$

3. *For all morphisms whose domains and codomains properly match*

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h) \tag{5}$$

4. *For all objects A,B we have*

$$1_A \otimes 1_B = 1_{A \otimes B}$$

---

[14] Hom-sets are not necessarily sets, but they are for small categories. In that case we more formally define $\text{hom}(-, -)$ as a covariant bifunctor (a kind of functor)

$$\text{hom}(-, -) : \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{Set}$$

This functor assigns pairs of domains and codomains (when there are morphisms between them) to particular sets of those morphisms.

Commentary: Part of what monoidal categories structure is an idea behind conjunction[15]

$$A \otimes B := \text{ system } A \textbf{ and } \text{system } B$$
$$f \otimes g := \text{ process } f \textbf{ and } \text{process } g$$

Monoidal categories are then able to structure categories that have both parallel and sequential processes such as **PhysProc**.

Note 6. We have defined a *strict* monoidal category here for convenience, but monoidal categories are not much different, though their definition is a bit more involved. Non-strict monoidal categories replace the equalities in the definition above definition with isomorphisms.[16]

At a basic level (strict) monoidal categories encode the structure underlying our naive notions of Time and Space when we look at simple categories like the following example:

Example 24. The category of cooking **Cook** has states of ingredients as objects (raw potato, cooked carrot, salted potato, etc.) and cooking processes as morphisms (do-nothing, boiling, frying, salting, slicing, etc.). Boiling is a morphisms $f : A \to B$ from raw potato to boiled potato. Salting is a morphism $g : B \to C$ from boiled potato to delicious potato. Slicing is a morphism $h : D \to E$ from raw carrot to sliced carrot. Frying is a morphism $j : E \to M$ from sliced carrot to crispy carrot. We can compose cooking processes in sequence, i.e.

$g \circ f$ boils the raw potato *then* salts the boiled potato.
$j \circ h$ slices the raw carrot *then* fries the sliced carrot.

Indeed **Cook** has the structure of a monoidal category, since we can not only cook sequentially, but also in parallel.

$1_{A \otimes D} : A \otimes D \to A \otimes D$ does nothing to a raw potato and a raw carrot.
$f \otimes h : A \otimes D \to B \otimes E$ boils the raw potato *while* slicing the raw carrot.

It is also true that this category satisfies Equation (5), which in this category says that

Boiling a potato *then* salting the potato **while** slicing a carrot *then* frying the carrot.
(is equivalent to)
Boiling a potato **while** slicing a carrot *then* salting the potato **while** frying the carrot.

[15] This is, however, more general than classical logical conjuction which requires that A AND A is equivalent to A, while in general $A \otimes A \neq A$. The more general logic that monoidal categories structure is referred to as linear logic.
[16] In fact, they replace them with what are called natural isomorphisms, another category theoretic idea.

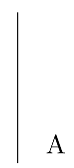
Naively we may want to attribute this to sequential processes being separated in time and parallel processes being separated in space, but this brings in physical notions of time and space where what we really want is the structure of two different processes with a certain relationship without accompanying metaphysical (or metaphorical) baggage, i.e. a monoidal category. We can then ask whether Time and Space do have some sort of monoidal categorical structure. This is then a question for physical experiment as our mathematics can structure whatever the results turn out to be. Can we define a category **Time** whose objects are things with some appropriate morphisms? It seems to be the case that it is better to think of spacetime, at least where relativity is concerned, but it is our hope that we have at least gestured toward how categories can structure such concepts.

5 Diagrammatic Categorical Reasoning

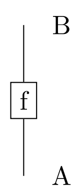
Categories, monoidal categories in particular have a rigorous graphical calculus, or "picturalism". This section won't try to be quite as technical, but will simply introduce the main ideas. Remember that underlying this (like there was a forgetful functor underlying the idea of quivers, but much more so) are some powerful theorems and structures that prove its rigor.

5.1 Graphical Calculus for Categories

Even this first definition of a category comes with a native graphical calculus. We represent an object A as an unbroken wire

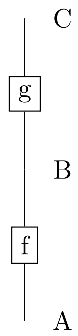


and a morphism $f : A \to B$ as boxes with objects (wires) as input and output



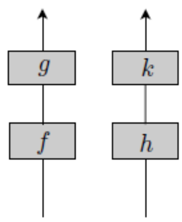
Note 7. We "read" these diagrams from bottom to top.

Composition is performed by attaching the next morphism box to the wire above the first, i.e. $g \circ f$ is diagrammed as

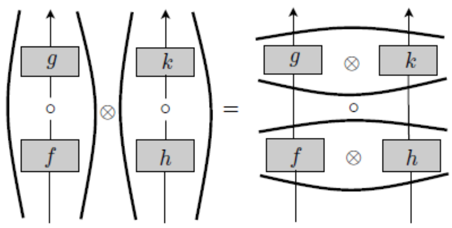


5.2 Graphical Calculus for Monoidal Categories

It is in application to monoidal categories that the graphical calculus really shines. By representing the monoidal operation as appending pictures horizontally on the plane, the structural isomorphisms of monoidal categories become intuitive. For example, the interchange law $(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$ (assuming the morphisms are well typed) is immediate as both sides of the equation are diagrammed as



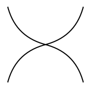
If we leave in some artificial brackets, this becomes more obvious



In general we have planar diagrams where horizontal composition is the tensor product and vertical composition is categorical composition. The monoidal identity I is left as a blank space on the page.

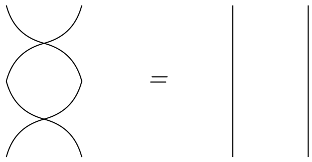
We will now present some other category theoretic ideas that can be presented in the graphical calculus (much more elegantly than in their traditional presentation), but we will present them quickly just to give a survey of the ways that category theoretic structures can be presented in this picturalism.

Definition 17. *The* **swap** *morphism* $\sigma : A \otimes B \to B \otimes A$ *is diagrammed as follows*



Commentary: It should be fairly obvious that this morphism structures our idea of swapping two things in space, which is why it is able to be rigorously presented as the crossing of two objects on the page.

Definition 18. *In a* **symmetric monoidal category** *the following graphical law is true*



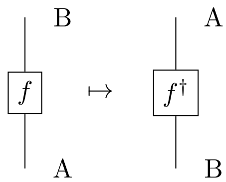
In symmetric monoidal categories, the swap morphism exists and $\sigma^{-1} = \sigma$*.*

Commentary: It does not have to be the case that swapping two items causes no irreversible changes. What if the monoidal product has the structural of rotations and reflections of an equilateral triangle in the plane? In that case the order that the operations are performed in does matter, so this would not be a symmetric monoidal category.[17]

Definition 19. *A* **dagger category** *is a category* $\mathbf{C}$ *equipped with a dagger functor* $\dagger : \mathbf{C} \to \mathbf{C}^{\mathbf{op}}$ *such that for all morphisms* f*:*

$$(f^\dagger)^\dagger = f$$

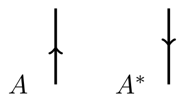
In the graphical calculus the dagger functor operation is to flip the diagram about its horizontal axis:



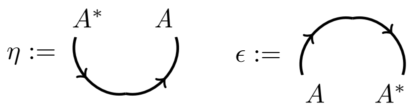
Commentary: This functor is an example of a kind of symmetry (perhaps causal symmetry) we could believe exists for sequential composition in a category.

[17] In general I suppose there is a relationship between symmetric monoidal categories and abelian groups that would apply here, but I'm not going to look it up or work it out right now.

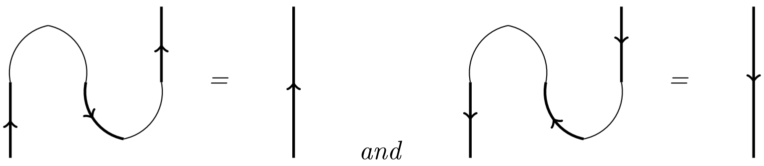
Definition 20. *The* **dual** A^* *to an object* A *in a symmetric monoidal category is drawn with arrows as follows*



such that the morphisms $\eta : I \to A^* \otimes A$ *and* $\epsilon : A \otimes A^* \to I$ *drawn as*



satisfy the following "snake equations":



Definition 21. *A* **compact category** *is a symmetric monoidal category for which every object has a dual.*

Commentary: In such a category we have the freedom to move boxes along wires, cross and uncross wires, bend wires and yank them straight. These categories structure our intuition for flows along wires in an idea that is fundamentally topological in nature.

Example 25. In the dagger compact category **FHilb** where

objects are finite dimensional Hilbert spaces
morphisms are finite linear maps
monoidal products are vector space tensor products
the unit $I = \mathbb{C}$
the dagger-functor the adjunction functor
all objects are self-dual

this graphical language has been extended into one that captures quantum mechanical structure as it relates to the finite dimensional Hilbert spaces manipulated in quantum computations. This is known as the Z/X calculus.

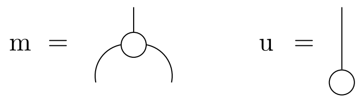
Commentary: What this says is that quantum mechanics has a structure of flows along wires. This is one way to rigorously speak of what we mean by flows of quantum information. The next section expands on how we know that it is information that is flowing.

6 Classical Information: Monoids in Categories

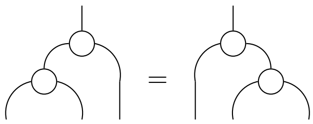
One structure necessary for our understanding of classical information, is that we can freely perform operations of copying and deleting. The structure of copying and deleting is that of monoids in monoidal categories.

Definition 22. *A* **monoid** *in a monoidal category $\mathcal{C}$ is a triple (A, m, u) of A in $Ob(\mathcal{C})$, a multiplication morphism $m : A \otimes A \to A$, and a unit $u : I \to A$ which satisfies associativity and unitality.*

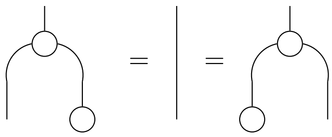
Diagrammatically this means that monoids have the following morphisms:



The requirements of associativity and unitality have the structure of the following graphical rules:

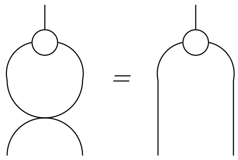


and



Commentary: Recall that monoids can be viewed as single object categories. This graphical definition simply says that when we compose the morphisms of a monoid we connect two wires (objects) to get a third in an associative way. This structure exists due to closure of composition. Secondly, it says that we have a unit morphism in the monoid so that whenever we combine it with anything else this is the same as doing nothing.

Definition 23. *A monoid is a* **commutative monoid** *when the following holds:*



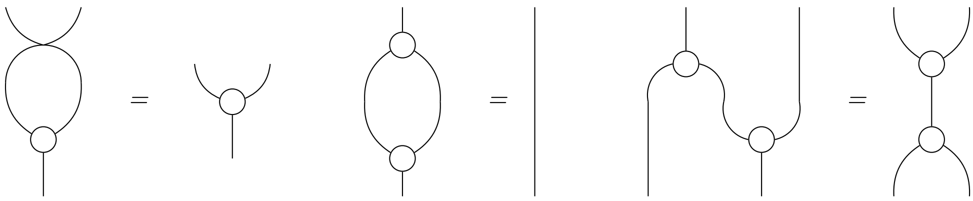
Commentary: This is the same as saying that the monoid is an Abelian category, i.e. the order of composition in the monoid category does not matter as $f \circ g = g \circ f$. If we then required that all the morphisms be invertible, the monoid would become an Abelian group.

Definition 24. *A* **comonoid** *in a monoidal category $\mathcal{C}$ is a triple (A, d, e) of $A \in Ob(\mathcal{C})$, a duplication morphism $d : A \to A \otimes A$, and a unit $e : A \to I$ which satisfies associativity and unitality.*

Graphically comonoids are represented by diagrams like those for the monoid, except flipped around a horizontal axis. This likewise gives the definition for a commutative comonoid.

By combining monoids and comonoids we are able to build axioms for classical flows of information. These are called classical structures.

Definition 25. *A* **classical structure** *in a dagger symmetric monoidal category is a commutative special dagger-Frobenius algebra. This is equivalent to a pair of a monoid and comonoid that satisfy the following three graphical laws:*



Commentary: The three graphical rules above structure our understanding that

1. Copies of (classical) information are indistinguishable, so we cannot tell if we swap them.
2. If we copy (classical) information and then compare the copies (keeping all that is the same), then we just get back what we started with.
3. From two copies of information, copying one and then comparing the copy (keeping all that is the same) to the third is the same as comparing the copies (keeping all that is the same) and then copying the result.

If we have all three of these rules, then we are freely able to copy information and delete redundant information. We note that these structures don't exist in every category, so ideas of computation in many categories does not come along with the ability to freely copy and delete information.

There have not been as many examples in this section because our aim has been to give a general flavor for what structures of category theory look like when embedded in our intuitions for boxes and wires on a plane. It may be easier to find and understand examples of these more complicated categorical structures once you have developed an intuition for their graphical calculus. And indeed, you can proceed confident that your intuitions for the graphical calculus are unlikely to lead you astray (at least in regard to certain categories like **Stab** for stabilizer quantum mechanics) where the graphical calculus has been shown to completely capture the structure of the theory. There is still much to be done in formalizing the rigor of the graphical calculus in more particular areas, but it will always be a helpful presentational tool. The innateness of these visual/diagrammatic presentations along with the ubiquity of categorical structure allows for the more or less rigorous visual presentation of structural connections between fields wherever they come up. This is the sort of thinking that has led to categorical diagrams of linguistic meaning, and could easily lead to the - again more or less rigorous - visual exploration of processes all over the place. What is the proper category for the dialectic turns in Hegel's *Phenomenology of Spirit*? Can we diagram their structure? What about the eschatological flavor in Marx's politics? How about the structure of social and anthropological relations? The narrative structure in the film *Memento*? Neural networks? Bayesian reasoning?

Further, we can turn things on their head and speak of the general structures of diagrams using category theory: flow charts, Feynman diagrams, spin foams in loop quantum gravity, electrical circuit diagrams, finite state automata, and, as always, more and more and more and more.

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2. Alan Weinstein. Groupoids: Unifying internal and external symmetry. *Notices of the American Mathematical Society*, 43(7):744–752, 1996.
3. I. M. Yaglom. *Felix Klein and Sophis Lie: Evolution of the idea of symmetry in the nineteenth century.* Birkhauser, 1988.

7 Further Developments

Some future notes about further developing this piece.

1. Lanham advises that a book by Gadamer could provide reference to differences in Category Theoretic structure between Kant and Hegel.
2. Another ripe example for continental categorification is deMan's writing on different conceptions of Time in poetry: metaphoric time, etc.
3. Consider introducing the diagrammatic language earlier.
4. Expand a whole section on duality theory.