Abstract structure of unitary oracles for quantum algorithms

William Zeng[1] Jamie Vicary[2]

[1]Department of Computer Science
University of Oxford

[2]Centre for Quantum Technologies, University of Singapore
and Department of Computer Science, University of Oxford

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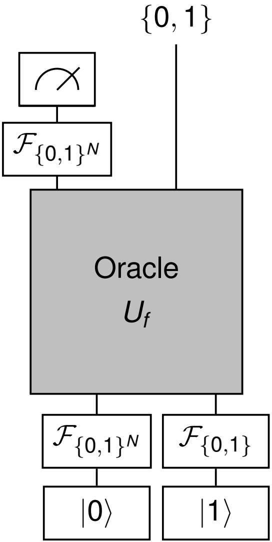
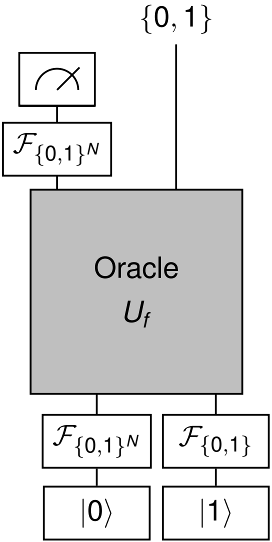
- Can we take advantage of this abstract setting to gain new insights?

Unitary Oracles

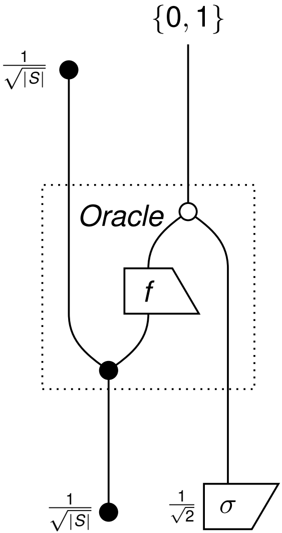
The traditional Deutsch-Joza circuit is:



Unitary Oracles

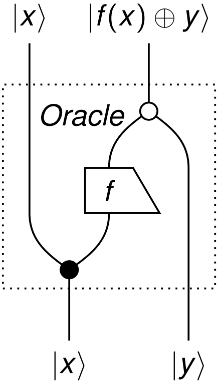


Here is its abstract structure:



Unitary Oracles

This is the oracle's internal structure:



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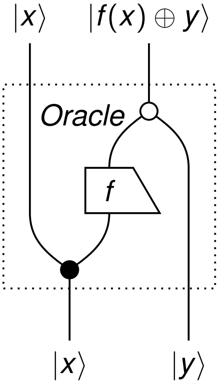
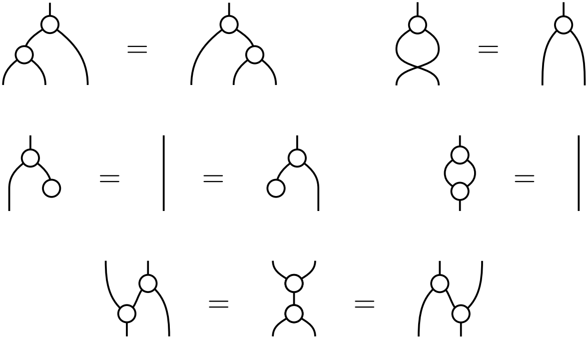


Theorem

Oracles with this abstract structure are unitary in general.

Categorical Quantum Information

Definition: A special †-Frobenius algebra obeys:



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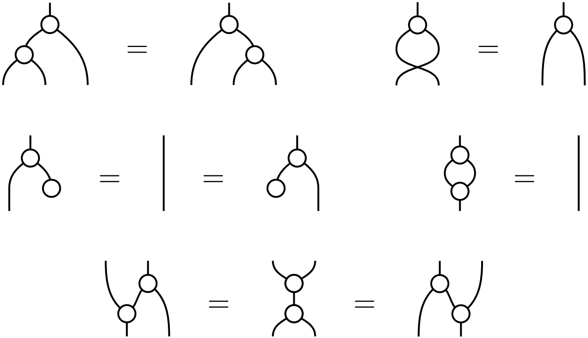
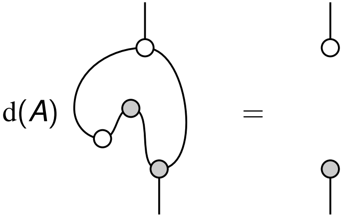


This represents the abstract structure of an *observable*.

Complementary observables

Definition [Coecke & Duncan]: Two †-Frobenius algebras on the same object are complementary when:



Complementary observables

Complementary observables in **FHilb** come from finite abelian groups

- Copying

 $:: |g\rangle \mapsto |g\rangle \otimes |g\rangle$

 $:: |g\rangle \mapsto 1$

- Group multiplication

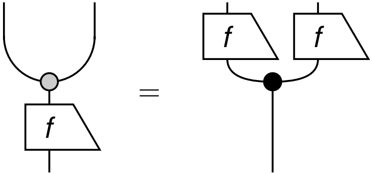
 $:: |g_1\rangle \otimes |g_2\rangle \mapsto \frac{1}{\sqrt{D}}|g_1 \oplus g_2\rangle$

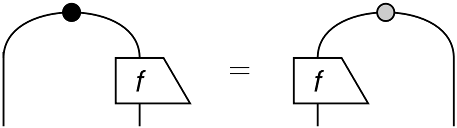
 $:: |1\rangle \mapsto \sqrt{D}|0\rangle$

Classical Maps

Definition:

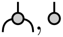
A classical map $f : (A, \text{black multiplication}, \text{black unit}) \to (B, \text{grey multiplication}, \text{grey unit})$ obeys:

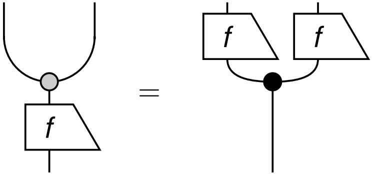


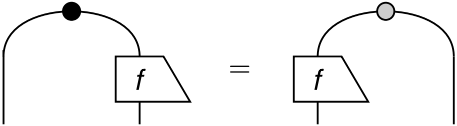


Classical Maps

Definition:

A classical map $f : (A,$  $) \to (B,$  $)$ obeys:





These are self-conjugate comonoid homomorphisms.

Unitarity Theorem

- Three †-Frobenius algebras, (◉ , ○, ●)

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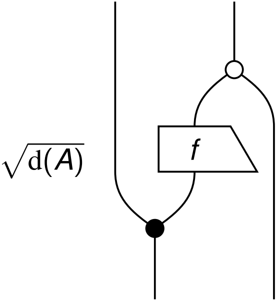
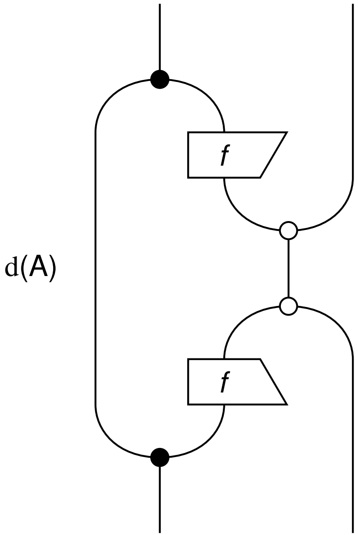
- A classical map $f : (A, ⅄●, ●) \to (B, ⅄◉, ◉)$

Produce the *unitary* morphism:



Abstract proof of unitarity



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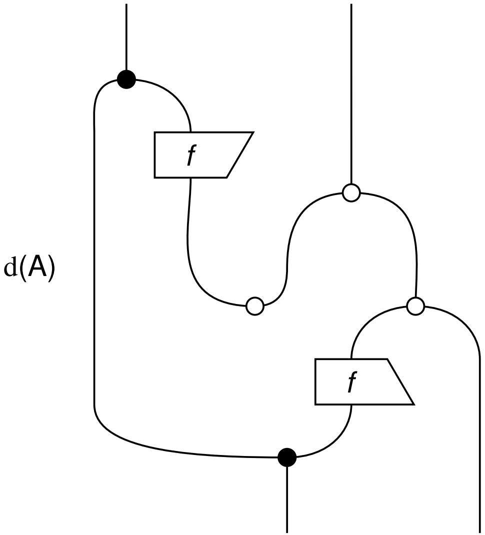

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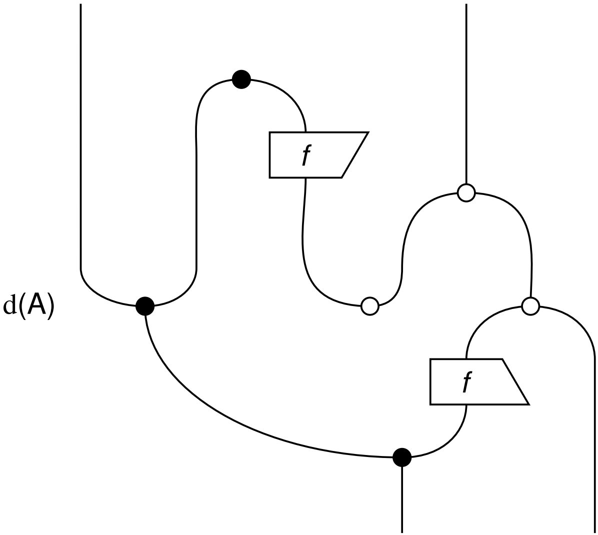

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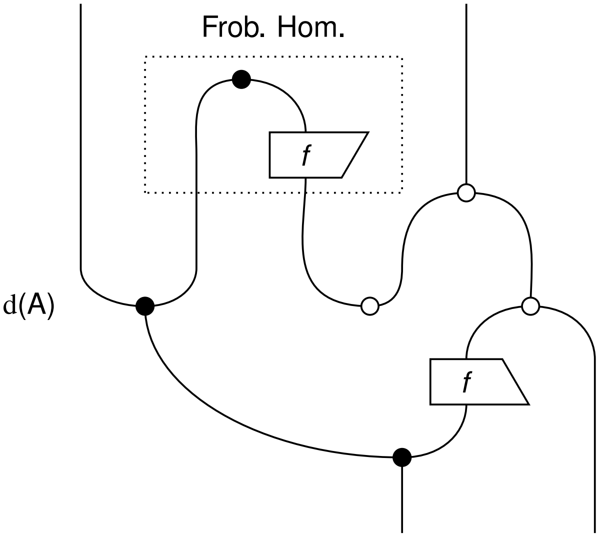

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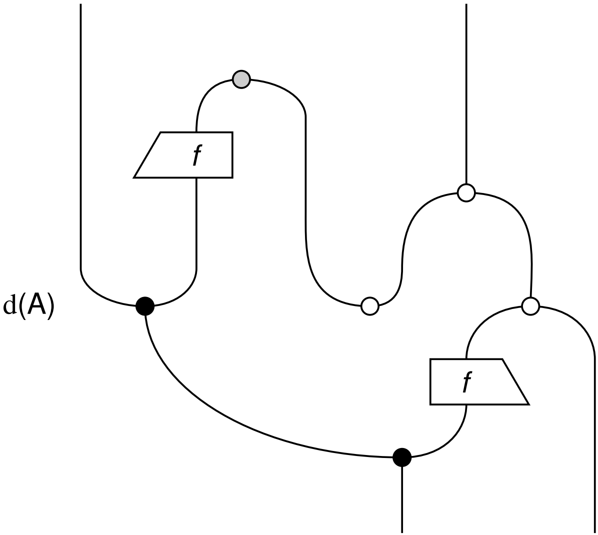

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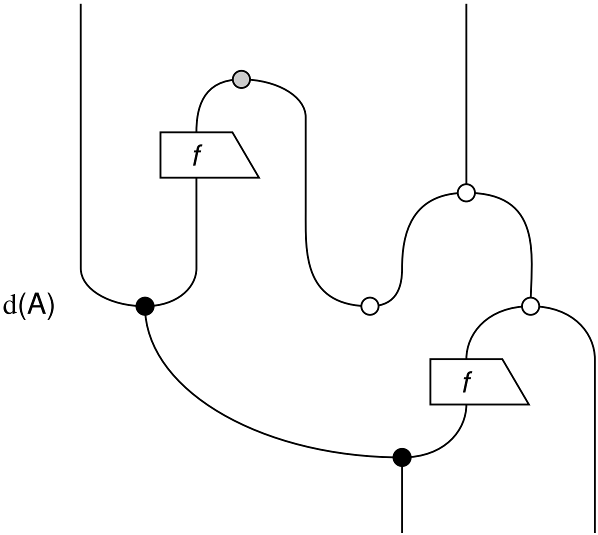

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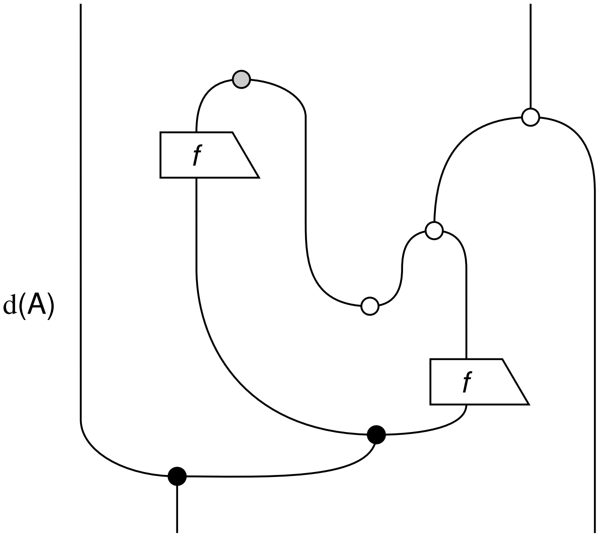

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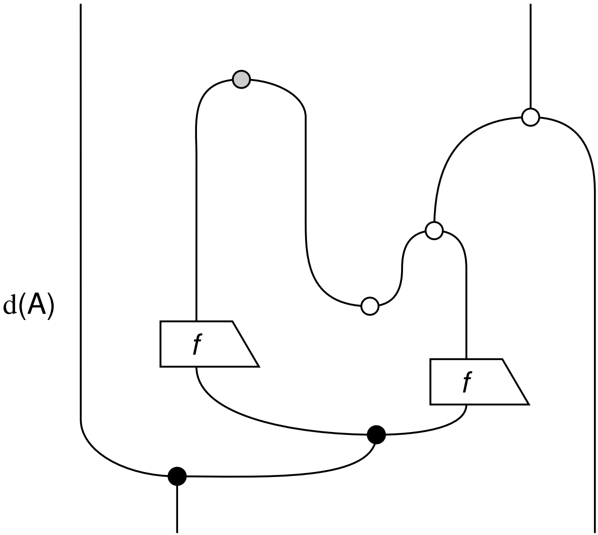

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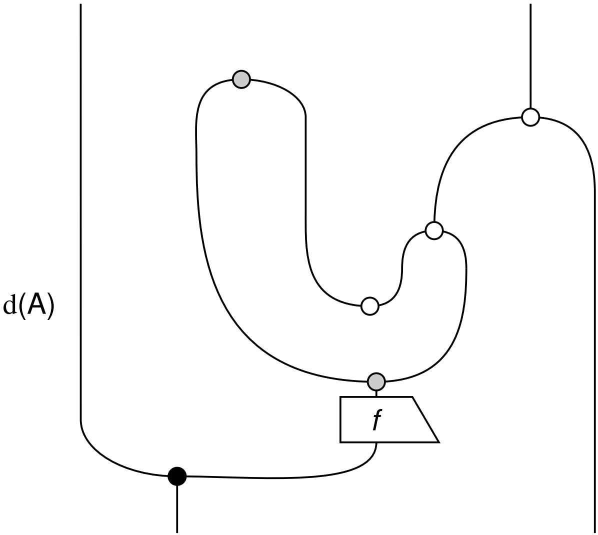

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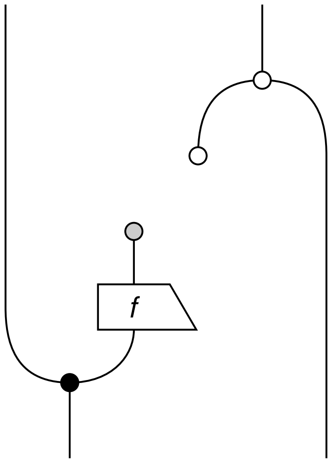

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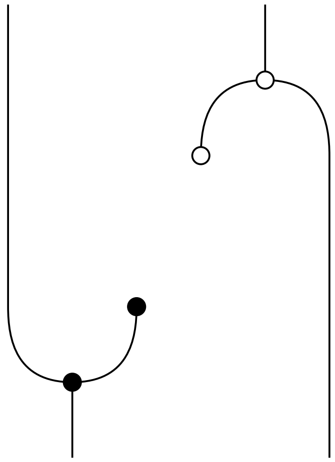

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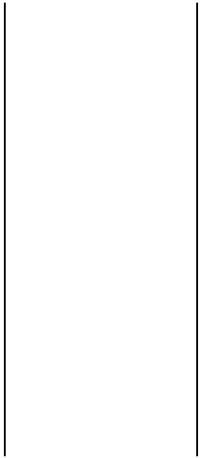

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- We have defined (diagrammatically) an abstract structure required to make oracles physical.
- This lifts the property of unitarity for quantum oracles to the more abstract setting of dagger monoidal categories.
- Can we take advantage of this abstract setting to gain new insights? Yes.
  - To develop a new group theoretic quantum algorithm
  - To apply result in signal-flow calculus

The group homomorphism identification problem

- Definition. (Group homomorphism identification problem) Given finite groups G and A where A is abelian, and a blackbox function $f : G \to A$ promised to be a group homomorphism, identify f.

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- Group Representations as measurements: projections onto a subspace

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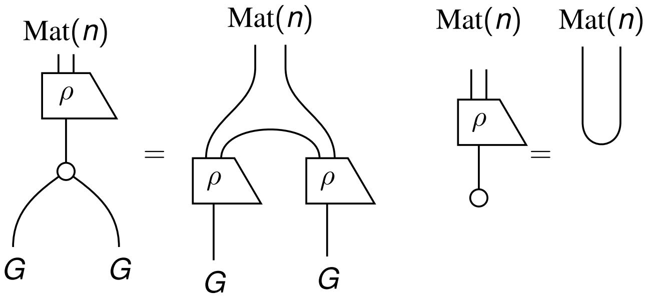
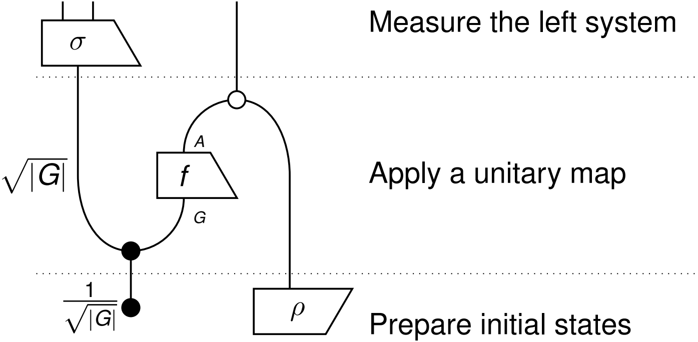
- Graphical rules for group representations:



The group homomorphism identification algorithm

Case: Let A be a cyclic group $\mathbb{Z}_n$.



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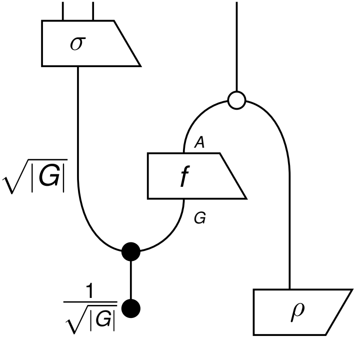

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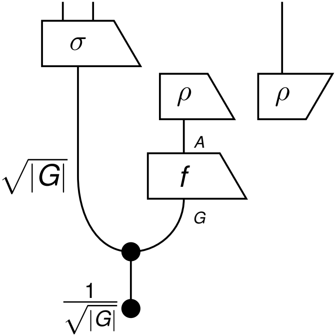

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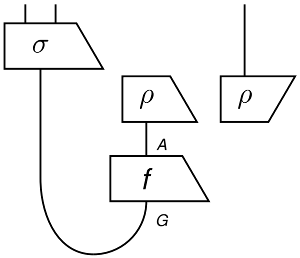

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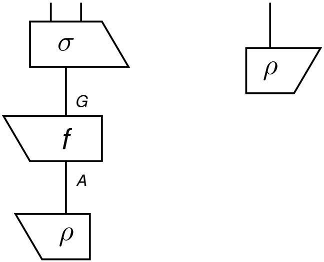

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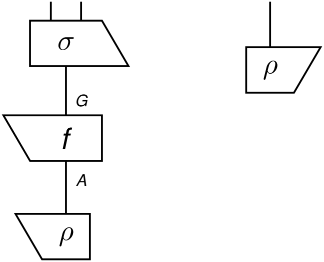

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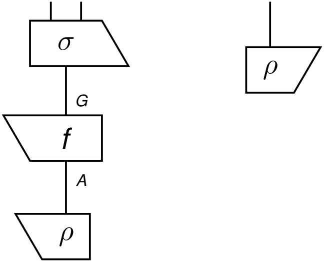

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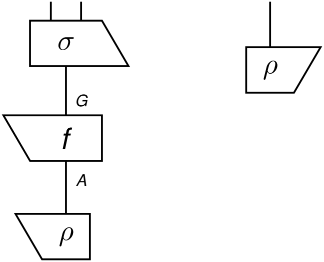

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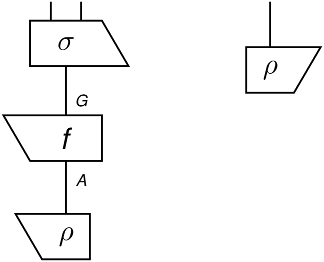

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- One-dimensional representations are isomorphic only if they are equal.

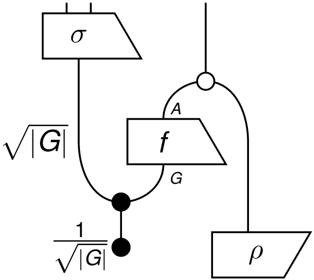
The group homomorphism identification algorithm

Homomorphism $f : G \to A$

- We generalize with proof by induction via the Structure Theorem. $A = Z_{p_1} \oplus ... \oplus Z_{p_k}$
- Can identify the group homomorphism in k oracle queries.
- The naive classical solution requires a number of queries equal to the number of factors of G rather than A.

Comparison to the hidden subgroup algorithm

Group ID



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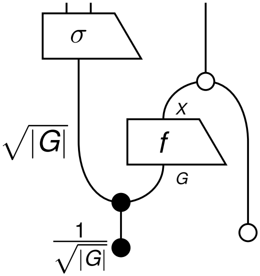

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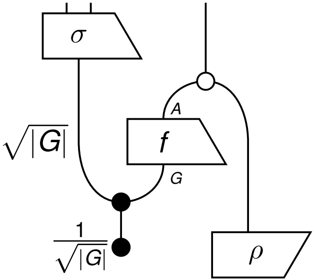

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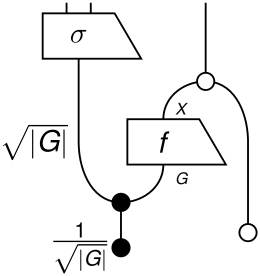

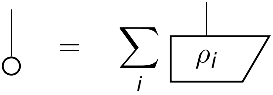

FinRel$_k$

[Baez, Erlebe, Fong 2014]

Definition

The category **FinRel**$_k$ of *linear relations* is defined in the following way, for any field k:

- **Objects** are finite dimensional k-vector spaces

$\mathbf{FinRel}_k$

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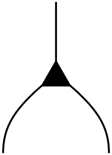
- **Objects** are finite dimensional k-vector spaces
- A **morphism** $f : V \to W$ is a *linear relation*, defined as a subspace $S_f \hookrightarrow V \oplus W$
- **Composition** of linear relations $f : U \to V$ and $g : V \to W$ is defined as the following subspace of $U \oplus W$:

$$\{(u, w) | \exists v \in V \text{ with } (u, v) \in S_f \text{ and } (v, w) \in S_g\}$$

  This defines a linear subspace of $U \oplus W$.

The signal-flow calculus $\mathbf{FinRel}_k$



Addition

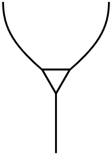
$\blacktriangle : k \oplus k \to k$

$(a, b, a+b) \in \blacktriangle$

Zero

$\bullet : \{0\} \to k$

$(0, 0) \in \bullet$



Copying

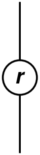
$\nabla : k \to k \oplus k$

$(a, a, a) \in \nabla$

Deletion

$\circ : k \to \{0\}$

$(a, 0) \in \circ$

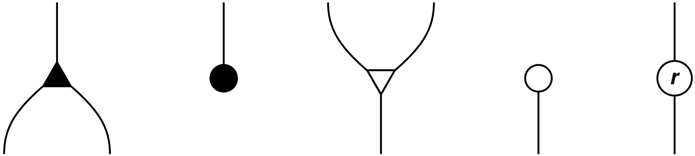


Multiplier

$\boldsymbol{r} : k \to k$

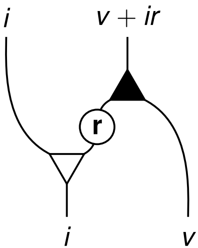
$(a, ra) \in \boldsymbol{r}$

The signal-flow calculus $\mathbf{FinRel}_k$



Addition	Zero	Copying	Deletion	Multiplier
$\blacktriangle : k \oplus k \to k$	$\bullet : \{0\} \to k$	$\triangledown : k \to k \oplus k$	$\circ : k \to \{0\}$	$\boldsymbol{r} : k \to k$
$(a, b, a+b) \in \blacktriangle$	$(0, 0) \in \bullet$	$(a, a, a) \in \triangledown$	$(a, 0) \in \circ$	$(a, ra) \in \boldsymbol{r}$

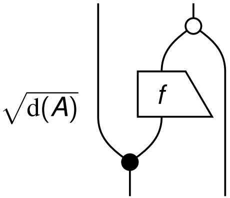
- Resistor



Conclusions (arxiv: 1406.1278)

- Theorem

  A pair of complementary dagger-Frobenius algebras, equipped with a classical map onto one of the algebras, produce a unitary morphism:



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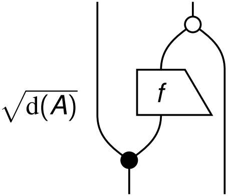

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  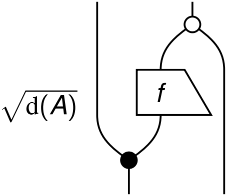

  - Abstract understanding of oracle in quantum computation
  - Apply this to develop a new algorithm for the deterministic identification of group homomorphisms into abelian groups.

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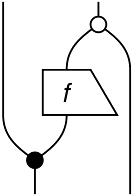
  $\sqrt{\mathrm{d}(A)}$ 

  - Abstract understanding of oracle in quantum computation
  - Apply this to develop a new algorithm for the deterministic identification of group homomorphisms into abelian groups.
  - Find the same structure in the theory of signal-flow networks.

Conclusions (arxiv: 1406.1278)

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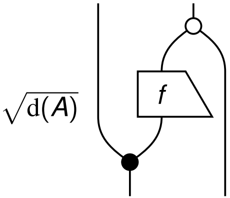

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  - Apply this to develop a new algorithm for the deterministic identification of group homomorphisms into abelian groups.
  - Find the same structure in the theory of signal-flow networks.
  - Big Idea: Symmetric monoidal categorical setting productively unifies process theories at an abstract level.

The Non-Abelian Case