Abstract structure of unitary oracles for quantum algorithms

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Quantum Physics and Logic, 2014
Unitary Oracles

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- What is the abstract structure of these oracles?
- Can we take advantage of this abstract setting to gain new insights?
Unitary Oracles

The traditional Deutsch-Joza circuit is:

\[ \begin{align*}
    |0, 1\rangle & \quad \xrightarrow{U_f} \quad |0, 1\rangle \\
    \mathcal{F}_{\{0,1\}^N} & \quad \text{Oracle} \quad \mathcal{F}_{\{0,1\}^N}
\end{align*} \]
Unitary Oracles

Here is its abstract structure:

\[ \frac{1}{\sqrt{|S|}} \]

Oracle

\[ f \]

\[ \frac{1}{\sqrt{2}} \]

\[ \sigma \]
Unitary Oracles

This is the oracle’s internal structure:

\[ |x\rangle, |f(x) \oplus y\rangle \]
Unitary Oracles

This is the oracle’s internal structure:

Oracle

Theorem

*Oracles with this abstract structure are unitary in general.*
Categorical Quantum Information

Definition: A special $\dagger$-Frobenius algebra obeys:

\[
\begin{align*}
\ 
\end{align*}
\]
Categorical Quantum Information

Definition: A special \(\dagger\)-Frobenius algebra obeys:

\[
\begin{align*}
\begin{array}{cccc}
\text{Diagram 1} & = & \text{Diagram 2} & = \\
\text{Diagram 3} & = & \text{Diagram 4} & = \\
\end{array}
\end{align*}
\]

This represents the abstract structure of an observable.
Definition [Coecke & Duncan]: Two $\dagger$-Frobenius algebras on the same object are \textit{complementary} when:

\[
\begin{align*}
\text{d}(A) & =
\end{align*}
\]
Complementary observables

Complementary observables in \textbf{FHilb} come from finite abelian groups

- **Copying**

  \[ \bigotimes \; : \; |g\rangle \mapsto |g\rangle \otimes |g\rangle \]

  \[ \mathbb{1} \; : \; |g\rangle \mapsto 1 \]

- **Group multiplication**

  \[ \bigotimes \; : \; |g_1\rangle \otimes |g_2\rangle \mapsto \frac{1}{\sqrt{D}} |g_1 \oplus g_2\rangle \]

  \[ \mathbb{0} \; : \; |1\rangle \mapsto \sqrt{D} |0\rangle \]
Definition:
A classical map \( f : (A, \odot, \bullet) \rightarrow (B, \odot, \circ) \) obeys:
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These are self-conjugate comonoid homomorphisms.
Unitarity Theorem

- Three $\dagger$-Frobenius algebras, $(\odot, \circ, \bullet)$
Unitarity Theorem

- Three $\dagger$-Frobenius algebras, ($\bigcirc$, $\circ$, $\bullet$)
- A pair are complementary ($\bigcirc$ and $\circ$)
Unitarity Theorem

- Three $\dagger$-Frobenius algebras, $(\circ, \circ, \bullet)$
- A pair are complementary ($\circ$ and $\circ$)
- A classical map $f : (A, \bullet, \bullet) \rightarrow (B, \wedge, \vee)$

Produce the *unitary* morphism:
Abstract proof of unitarity

d(A)

f

f
Abstract proof of unitarity

d(A)

\[ f \]

\[ f \]
Abstract proof of unitarity
Abstract proof of unitarity

Frob. Hom.

$d(A)$

$f$

$f$
Abstract proof of unitarity
Abstract proof of unitarity
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Unitary Oracles

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Unitary Oracles

- We have defined (diagrammatically) an abstract structure required to make oracles physical.
- This lifts the property of unitarity for quantum oracles to the more abstract setting of dagger monoidal categories.
- Can we take advantage of this abstract setting to gain new insights? Yes.
  - To develop a new group theoretic quantum algorithm
  - To apply result in signal-flow calculus
Definition. (Group homomorphism identification problem)
Given finite groups $G$ and $A$ where $A$ is abelian, and a blackbox function $f : G \rightarrow A$ promised to be a group homomorphism, identify $f$. 
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**Group representations are** $\rho : G \to \text{Mat}(n)$

<table>
<thead>
<tr>
<th>Abelian</th>
<th>Non-abelian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>$\text{Mat}(n)$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\rho$</td>
</tr>
</tbody>
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The group homomorphism identification problem

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- Group representations are $\rho : G \rightarrow \text{Mat}(n)$

  ![Abelian Group Representation](image)
  ![Non-abelian Group Representation](image)

- Group Representations as measurements: projections onto a subspace
The group homomorphism identification problem

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- **Graphical rules for group representations:**

  ![Graphical rules for group representations](image-url)
The group homomorphism identification algorithm

Case: Let $A$ be a cyclic group $\mathbb{Z}_n$.

- Prepare initial states
- Apply a unitary map
- Measure the left system

\[
\frac{1}{\sqrt{|G|}}
\]

\[
\sqrt{|G|}
\]
The group homomorphism identification algorithm
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The group homomorphism identification algorithm
The group homomorphism identification algorithm

\[ \sigma \xrightarrow{f} G \xrightarrow{A} \rho \]
The group homomorphism identification algorithm

\[ \sigma \xrightarrow{f} G \xrightarrow{\rho} \rho \circ f \text{ is an irreducible representation of } A. \]
The group homomorphism identification algorithm

- \( \rho \circ f \) is an irreducible representation of \( A \).
- Choose \( \rho \) to be a faithful representation of \( A \).
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- Choose \( \rho \) to be a faithful representation of \( A \).
- Then measuring \( \rho \circ f \) identifies \( f \) (up to isomorphism).
- One-dimensional representations are isomorphic only if they are equal.
The group homomorphism identification algorithm

Homomorphism $f : G \rightarrow A$

- We generalize with proof by induction via the Structure Theorem. $A = \mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_k}$
- Can identify the group homomorphism in $k$ oracle queries.
- The naive classical solution requires a number of queries equal to the number of factors of $G$ rather than $A$. 
Comparison to the hidden subgroup algorithm

Group ID

Hidden Subgroup

\[ \sqrt{|G|} \]

\[ \frac{1}{\sqrt{|G|}} \]

\( \sigma \)

\( A \)

\( f \)

\( G \)

\( \rho \)

\[ \sqrt{|G|} \]

\[ \frac{1}{\sqrt{|G|}} \]

\( \sigma \)

\( X \)

\( f \)

\( G \)
Comparison to the hidden subgroup algorithm

Group ID

\[
\sqrt{|G|} \quad f \quad \frac{1}{\sqrt{|G|}} \quad \rho
\]

Hidden Subgroup

\[
\sqrt{|G|} \quad f \quad \frac{1}{\sqrt{|G|}} \quad \rho
\]

\[
= \sum_{i} \rho_i
\]
**Definition**

The category $\text{FinRel}_k$ of *linear relations* is defined in the following way, for any field $k$:

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- **Objects** are finite dimensional \( k \)-vector spaces
- A **morphism** \( f : V \to W \) is a \textit{linear relation}, defined as a subspace \( S_f \hookrightarrow V \oplus W \)
- **Composition** of linear relations \( f : U \to V \) and \( g : V \to W \) is defined as the following subspace of \( U \oplus W \):

\[
\{(u, w) | \exists v \in V \text{ with } (u, v) \in S_f \text{ and } (v, w) \in S_g \}
\]

This defines a linear subspace of \( U \oplus W \).
The signal-flow calculus $\text{FinRel}_k$

- **Addition**: $\Delta : k \oplus k \to k$
  
  $(a, b, a + b) \in \Delta$

- **Zero**: $\bullet : \{0\} \to k$
  
  $(0, 0) \in \bullet$

- **Copying**: $\nabla : k \to k \oplus k$
  
  $(a, a, a) \in \nabla$

- **Deletion**: $\bigcirc : k \to \{0\}$
  
  $(a, 0) \in \bigcirc$

- **Multiplier**: $r : k \to k$
  
  $(a, ra) \in r$
The signal-flow calculus $\text{FinRel}_k$

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- **Multiplier**: $\bullet_r: k \rightarrow k$
  
  $(a, ra) \in \bullet_r$

- **Resistor**

  $i$  
  
  $v + ir$

  $r$

  $i$

  $v$
Theorem

A pair of complementary dagger-Frobenius algebras, equipped with a classical map onto one of the algebras, produce a unitary morphism:
Conclusions (arxiv: 1406.1278)

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  $\sqrt{d(A)}$

  ![Diagram](image)

  - Abstract understanding of oracle in quantum computation
  - Apply this to develop a new algorithm for the deterministic identification of group homomorphisms into abelian groups.
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  ![Diagram](image)

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- Find the same structure in the theory of signal-flow networks.
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▶ Theorem

A pair of complementary dagger-Frobenius algebras, equipped with a classical map onto one of the algebras, produce a unitary morphism:

\[ \sqrt{d(A)} f \]

▶ Abstract understanding of oracle in quantum computation

▶ Apply this to develop a new algorithm for the deterministic identification of group homomorphisms into abelian groups.

▶ Find the same structure in the theory of signal-flow networks.

▶ Big Idea: Symmetric monoidal categorical setting productively unifies process theories at an abstract level.
The Non-Abelian Case

\[
\sqrt{|G|} \sigma \rho^\dagger \psi = \rho^\dagger \rho \psi
\]